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Counting Patterns in Graphs*

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The main problem is that of counting partial subgraphs (or patterns, for short) in a given graph by inspection of the complementary graph (indirect method). As an illustration of the general approach described, specific formulas are derived for Hamiltonian circuits and complete subgraphs of any order.

1. INTRODUCTION

Let $G = (X, U)$ be a finite graph (when no mention is made to the contrary, every graph-theoretic term employed in this paper must be understood in Berge's sense [1]), that is, $X = V(G)$ is the set of vertices and $U = W(G)$ the set of links. Hereafter, links will be called arcs or edges according to whether they are oriented or not. The complete symmetric graph (oriented or not, according with G) defined on X will be denoted by $G_n = (X, U_n)$, $n = |X|$, while $\bar{G} = (X, \bar{U})$, $\bar{U} = U_n - U$, will be referred to as the complementary graph of G. Recall that the power $P(S)$ of a set S is the set of all subsets of S; the set of all proper subsets of S will be called the proper power and denoted $P'(S)$. A subset of S with k elements is often called a k-subset; the set of all k-subsets will be called the k-power of S and denoted $P^k(S)$.

For any graph $G = (X, U)$, $P(U)$ will be called the link-power of G and denoted $P_w(G)$. The link-power of a set Δ of graphs is defined to be the union of the link-powers of the graphs in Δ. Proper link-powers are defined similarly.

Some other terminology and notations are required for a statement of our general theorem: Let Γ be a properly defined set of patterns of G_n, e.g., complete subgraphs, regular partial subgraphs, paths, chains, circuits, cycles, paths, or chains between two fixed vertices. The set Γ

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of graphs will be said to consist of relatively prime graphs if none of its subgraphs is a partial subgraph of another.

Graphs of Γ which are patterns of G will also be called γ-patterns of G. The set of all γ-patterns of G will be denoted by Γ_G. More specifically, $\Gamma_G{}^e$ and $\Gamma_G{}^0$ will indicate the subsets of Γ_G consisting of those γ-patterns of G which have an even and an odd number of links, respectively. Moreover, for any $S \subseteq U_n$, Γ^S is the subset of Γ consisting of graphs covering S, i.e.,

$$\Gamma^S = \{\gamma / \gamma \in \Gamma,\ W(\gamma) \supseteq S\}.$$

The following theorem can now be stated [2, 3]:

THEOREM 1. *The number of γ-patterns of G is given by:*

$$|\Gamma_G| = |\Gamma| + \sum_{k=1}^{K(\bar{U},\Gamma)} (-1)^k \sum_{S \in R_k(\bar{U},\Gamma)} |\Gamma^S|,$$

where $R_k(\bar{U}, \Gamma) = P^k(\bar{U}) \cap P_w(\Gamma)$ and $K(\bar{U}, \Gamma)$ is the minimum value of k such that $R_{k+1}(\bar{U}, \Gamma) = \varnothing$.

There is no difficulty in seeing that, if $R_k(\bar{U}, \Gamma) = \varnothing$, then $R_{k+i}(\bar{U}, \Gamma) = \varnothing$, for all positive integers i.

COROLLARY 1. *If Γ is a set of prime graphs, then*

$$|\Gamma_G| = |\Gamma_G^e| - |\Gamma_G^0| + |\Gamma| + \sum_{k=1}^{K'(\bar{U},\Gamma)} (-1)^k \sum_{S \in R_k'(\bar{U},\Gamma)} |\Gamma^S|,$$

where $R_k'(\bar{U}, \Gamma) = P^k(\bar{U}) \cap P_w'(\Gamma)$ and $K'(\bar{U}, \Gamma)$ is the minimum value of k such that $R'_{k+1}(\bar{U}, \Gamma) = \varnothing$.

Theorem 1 and Corollary 1 relate a single counting problem defined in G to many different counting problems defined in G_n. However, it must be pointed out that $|\Gamma^S|$ can often be very simply computed once and for all, whatever S and n may be, provided that the uniform structure of G_n is taken into account, as will be shown later by means of examples.

This paper has been organized in sections as follows. The problem of counting Hamiltonian circuits in arbitrary oriented graphs is faced in Section 2. Sections 3 and 4 deal with counting complete subgraphs (of any order) in arbitrary non-oriented graphs. Some concluding remarks may be found at the end of the paper.

2. Counting Hamiltonian Circuits

Let G be an oriented graph and Φ be the set of all possible Hamiltonian circuits of G_n. Hamiltonian circuits are prime graphs; then, from Corollary 1, it follows that

$$| \Phi_G | = (-1)^n | \Phi_{\bar{G}} | + | \Phi | + \sum_{k=1}^{K'(\bar{U},\Phi)} (-1)^k \sum_{S \in R'(\bar{U},\Phi)} | \Phi^S |, \quad (1)$$

as there are n arcs in any Hamiltonian circuit of $\bar{G}$. A general formula for $| \Phi^S |$ will be given by Theorem 2, but, in order to simplify the proof of that theorem, it is convenient to start with the following lemma, which is given without proof.

Lemma. *Let $G = (X, U)$ be a graph the components of which are elementary paths or isolated vertices. Then the number of the (connected) components of G is given by the difference of the cardinalities of X and U.*

Theorem 2. *For any set $S \in P_W'(\Phi)$, $| S | = k < n$,*

$$| \Phi^S | = (n - k - 1)!$$

Proof. Note that $G_s = (X, S)$, $S \in P_W'(\Phi)$, is always a partial graph of a Hamiltonian circuit of G_n. Therefore, in view of the lemma above, G_s has $n - k$ connected components. Since the number of Hamiltonian circuits of G_n meeting with all the arcs of S is just the number of different ways in which the $n - k$ connected components of G_s can be joined in a closed sequence, then $| \Phi^S |$ equals the number of distinct permutations of $n - k - 1$ elements, i.e., $(n - k - 1)!$.

Corollary 2. *Let $\rho_k(\bar{U}, \Phi) = | R_k'(\bar{U}, \Phi)|$. Then $\rho_1(\bar{U}, \Phi) = | \bar{U} |$ and, in view of Theorem 2, equation (1) can be rewritten as:*

$$| \Phi_G | = (-1)^n | \Phi_{\bar{G}} | + (n - 1)! + \sum_{k=1}^{K'(\bar{U},\Phi)} (n - k - 1)! \, (-1)^k \rho_k(\bar{U}, \Phi).$$

This equation allows the elementary circuits in G to be counted by inspection of the complementary graph $\bar{G}$. It is worth noticing that a counting formula for Hamiltonian circuits, based on the inclusion-exclusion principle, has appeared in [4]. The main difference between Wilf's use of the inclusion-exclusion principle and the present one rests on the definition of "properties" (meeting or not with vertices of G, instead of meeting or not with links of $\bar{G}$) so that, from a computational point of view, only the present method is an indirect one based on the analysis of the complementary graph.

Examples. Let α (Fig. 1a) be the graph of which the Hamiltonian circuits are to be counted. Its complementary graph $\bar{\alpha}$ is shown in Fig. 1b. Vertices are numbered 1, 2, 3, 4.

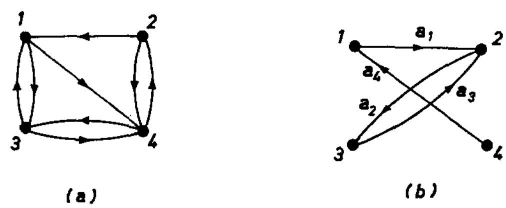

FIG. 1. (a) The graph α. (b) The complementary graph $\bar{\alpha}$.

A detailed analysis of $\bar{\alpha}$ yields:

$$
\begin{aligned}
W(\bar{\alpha}) &= \{(1,2),(2,3),(3,2),(4,1)\} = \{a_1, a_2, a_3, a_4\} = A,\\
\Phi_{\bar{\alpha}} &= \varnothing,\\
R_2'(A, \Phi) &= \{\{a_1, a_2\}, \{a_1, a_4\}, \{a_2, a_4\}, \{a_3, a_4\}\},\\
R_3'(A, \Phi) &= \{\{a_1, a_2, a_4\}\},\\
R_4'(A, \Phi) &= \varnothing.
\end{aligned}
$$

Then

$$K'(A, \Phi) = 3, \qquad \rho_1(A, \Phi) = \rho_2(A, \Phi) = 4, \qquad \rho_3(A, \Phi) = 1,$$

and

$$|\,\Phi_\alpha\,| = 0 + 3! - 2!\,4 + 1!\,4 - 0!\,1 = 1.$$

This is a trivial result, as it is quite evident that 12431 is the only Hamiltonian circuit of α. But, if the graph β of Fig. 2 is now considered, it is easy to see that $W(\bar{\beta}) = W(\bar{\alpha}) = A$.

Therefore, the preceding analysis holds and, from Corollary 2, it follows that:

$$|\,\Phi_\beta\,| = 0 + 6! - 5!\,4 + 4!\,4 - 3!\,1 = 330.$$

Due to the "quasi-completeness" of β, a direct analysis of the graph would have required cumbersome computations by-passed here by the previously described indirect method.

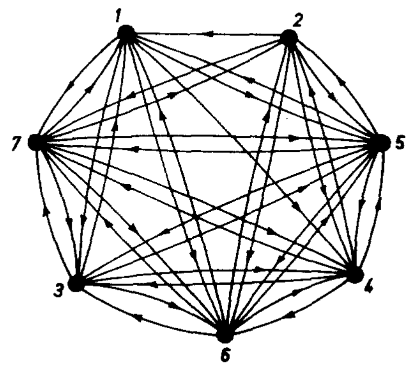

FIG. 2. The graph β.

3. COUNTING COMPLETE SUBGRAPHS

A complete subgraph of order j of a given non-oriented graph G will be called here (j)-subgraph of G, for the sake of conciseness. Since (j)-subgraphs are prime patterns, Corollary 1 can still be employed; hence, letting Ω^j be the set of all possible (j)-subgraphs of G_n,

$$|\Omega_G{}^j| = (-1)^{C_{j,2}} |\Omega_{\bar{G}}{}^j| + |\Omega^j| + \sum_{k=1}^{K'(\bar{U},\Omega^j)} (-1)^k \sum_{S \in R_k'(\bar{U},\Omega^j)} |\Omega^{jS}|, \tag{3}$$

where $C_{r,s} = \binom{r}{s}$; hence $C_{j,2}$ is the number of edges of a (j)-subgraph of $\bar{G}$.

THEOREM 3. *For any set $S \in P_W'(\Omega^j)$, $|S| = k < n$,*

$$|\Omega^{jS}| = C_{n-m,j-m}, \tag{4}$$

where m is the number of distinct vertices meeting edges of S.

Proof. As a matter of fact, for any given set S, Ω^{jS} can easily be constructed by choosing in all possible ways $j - m$ of the $n - m$ vertices of X which do not meet with any edge of S. Thus the cardinality of Ω^{jS} is just the number of combinations of class $j - m$ of $n - m$ elements, namely, $C_{n-m,j-m}$.

It is convenient to introduce here a more specific notation. Let $R_k'(\bar{U}, \Omega^j, m)$ be the subset of $R_k'(\bar{U}, \Omega^j)$ consisting of all the subsets S of $\bar{U}$ meeting with exactly m vertices of X. Then, there is no difficulty in seeing that

$$|R_1'(U, \Omega^j, 2)| = |\bar{U}|. \tag{5}$$

Furthermore, let

$$\rho(\overline{U}, \Omega^j, m) = \sum_{k=2}^{K'(\overline{U},\Omega^j)} (-1)^k \mid R_k'(\overline{U}, \Omega^j, m)|, \qquad m = 3, 4,..., j. \quad (6)$$

COROLLARY 3. *In view of Theorem* 3, *with equations* (5) *and* (6) *taken into account, equation* (3) *can be rewritten*:

$$| \Omega_G{}^j | = (-1)^{C_{j,2}} | \Omega_{\bar{G}}{}^j | + C_{n,j} - | \overline{U} | C_{n,j} - | \overline{U} | C_{n-2,j-2} + \sum_{m=3}^{j} C_{n-m,j-m}\rho(\overline{U}, \Omega^j, m).$$

Proof. In view of Theorem 3, it turns out that

$$\sum_{k=1}^{K'(\overline{U},\Omega^j)} (-1)^k \sum_{S\in R_k'(\overline{U},\Omega^j)} | \Omega^{jS} | = \sum_{k=1}^{K'(\overline{U},\Omega^j)} (-1)^k \sum_{m=2}^{j} C_{n-m,j-m} \mid R_k'(\overline{U}, \Omega^j, m)|.$$

Then, in view of equation (5),

$$\sum_{m=2}^{j} C_{n-m,j-m} \sum_{k=1}^{K'(\overline{U},\Omega^j)} (-1)^k \mid R_k'(\overline{U}, \Omega^j, m)| = -| \overline{U} | C_{n-2,j-2} + \sum_{m=3}^{j} C_{n-m,j-m}\rho(\overline{U}, \Omega^j, m),$$

where definition (6) has also been used.

Example. The (5)-subgraphs of the graph η, shown in Fig. 3a, will be counted here by means of Corollary 3. Vertices are numbered 1, 2,..., 8. A detailed analysis of the complementary graph (Fig. 3b) yields:

$$\begin{aligned}
W(\bar{\eta}) &= \{(1, 7), (2, 4), (2, 5), (4, 5), (4, 6)\} \\
&= \{b_1, b_2, b_3, b_4, b_5\} = B, \\
R_2'(B, \Omega^5, 3) &= \{\{b_2, b_3\}, \{b_2, b_4\}, \{b_2, b_5\}, \{b_3, b_4\}, \{b_4, b_5\}\}, \\
R_2'(B, \Omega^5, 4) &= \{\{b_1, b_2\}, \{b_1, b_3\}, \{b_1, b_4\}, \{b_1, b_5\}, \{b_3, b_5\}\}, \\
R_3'(B, \Omega^5, 3) &= \{\{b_2, b_3, b_4\}\}, \\
R_3'(B, \Omega^5, 4) &= \{\{b_2, b_3, b_5\}, \{b_2, b_4, b_5\}, \{b_3, b_4, b_5\}\}, \\
R_3'(B, \Omega^5, 5) &= \{\{b_1, b_2, b_3\}, \{b_1, b_2, b_4\}, \{b_1, b_2, b_5\}, \{b_1, b_3, b_4\}, \\
&\qquad \{b_1, b_4, b_5\}\},
\end{aligned}$$

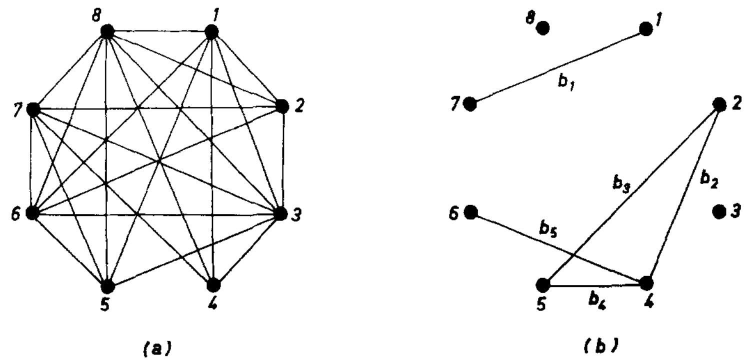

FIG. 3. (a) The graph η. (b) The complementary graph $\bar{\eta}$.

$$
\begin{aligned}
R_4'(B, \Omega^5, 4) &= \{\{b_2, b_3, b_4, b_5\}\},\\
R_5'(B, \Omega^5, 5) &= \{\{b_1, b_2, b_3, b_4\}\},\\
R_5'(B, \Omega^5, m) &= \varnothing, \qquad m = 4, 5.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
K'(B, \Omega^5) &= 4,\\
\rho(B, \Omega^5, 3) &= 5 - 1 = 4,\\
\rho(B, \Omega^5, 4) &= 5 - 3 + 1 = 3,\\
\rho(B, \Omega^5, 5) &= -5 + 1 = -4,
\end{aligned}
$$

and

$$
| \Omega_\eta^5 | = 0 + 56 - 5 \cdot 20 + 10 \cdot 4 + 4 \cdot 3 - 1 \cdot 4 = 4
$$

is the required result.

5. Concluding Remarks

A new way of counting patterns in graphs by inspection of the complementary graph (indirect method) has been used in this paper to count Hamiltonian circuits in oriented graphs and complete subgraphs of any order in non-oriented graphs. Formulae for the number of spanning trees in non-oriented graphs may be found in [5], while regular partial subgraphs of order 6 and degree 3, of the Kuratowski type, are also counted in [2]. This, together with [3], completes the picture of the applications until now available of the indirect method discussed, to some extent, at the beginning of the present paper.

References

1. C. Berge, "The Theory of Graphs and Its Applications," Methuen, London/Wiley, New York, 1962.
2. G. Guardabassi, Counting patterns in graphs: a necessary planarity condition, Internal Report LE 66-6, Istituto di Elettrotecnica ed Elettronica del Politecnico di Milano, Milano (Italy), 1966; presented at the International Seminar on Graph Theory, Rome, July 1966.
3. E. Biondi, L. Divieti, and G. Guardabassi, Counting paths, circuits, chains and cycles in graphs: a unified approach, *Canad. J. Math.* **22** (1970), 22–35.
4. H. S. Wilf, A mechanical counting method and combinatorial applications, *J. Combinatorial Theory* **4** (1968), 246–258.
5. G. Guardabassi, Indirect counting trees in linear graphs, *J. Franklin Inst.* **291** (1971), 101–107.